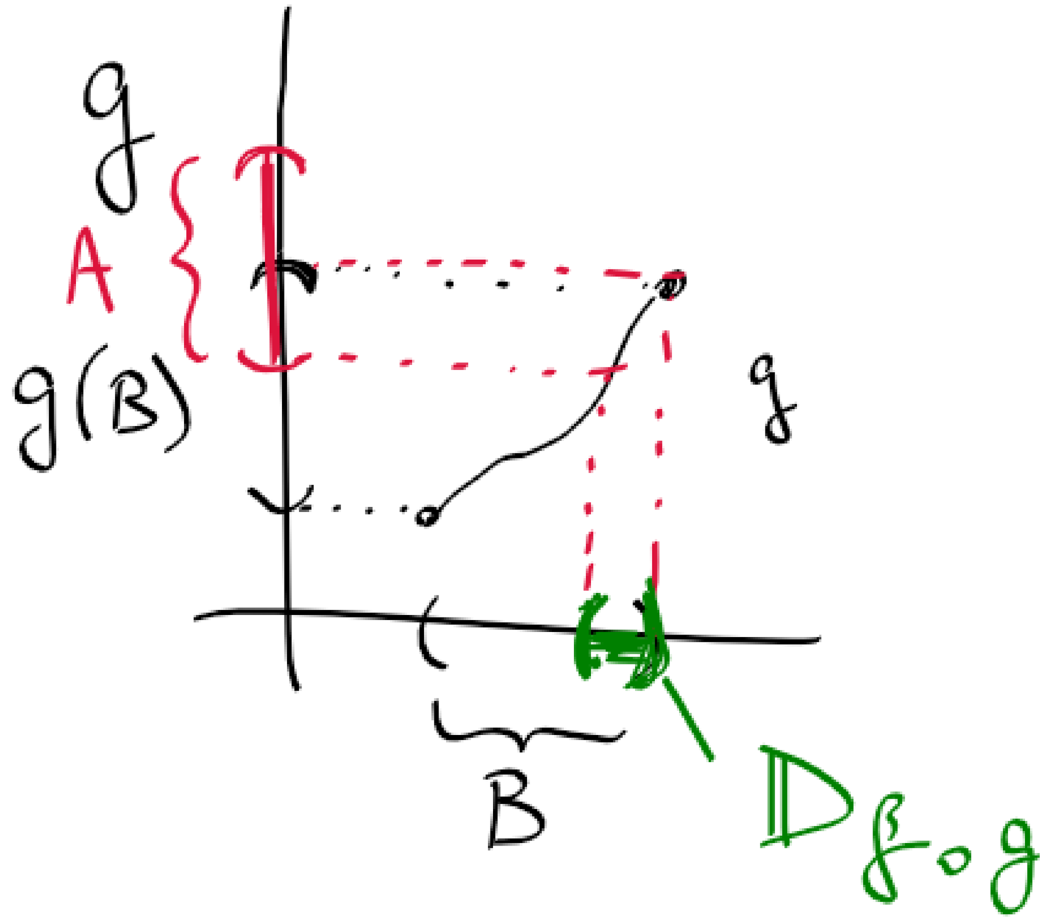
Skládání funkcí více proměnných

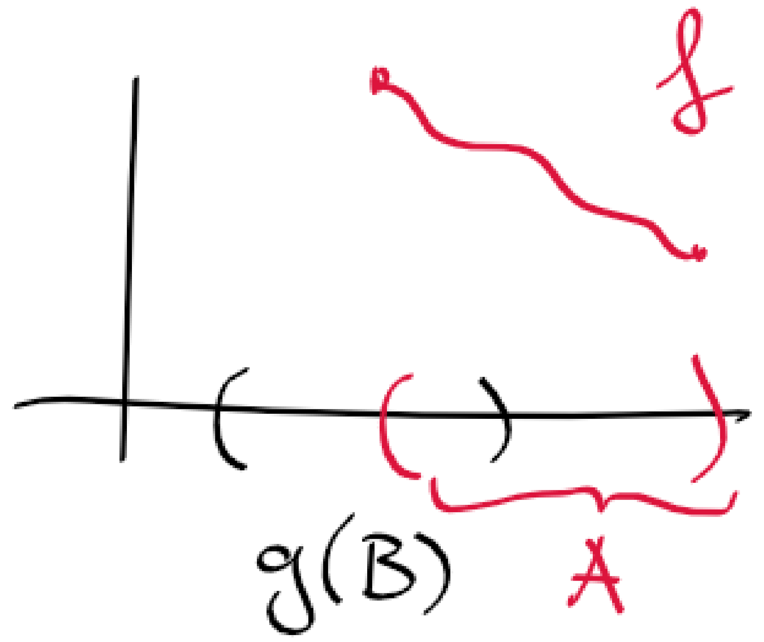
Připomeňme: $A, B \subseteq \mathbb{R}$

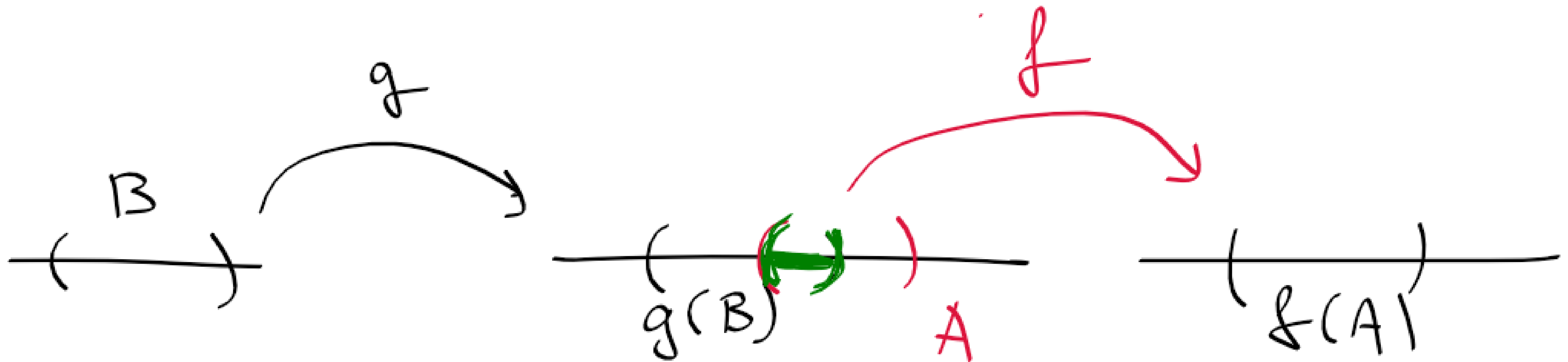
$f: A \to \mathbb{R}$, $g: B \to \mathbb{R}$

Složení $f \circ g\,(x) = f(\underbrace{g(x)}_{\overset{?}{\in} \mathbb{D}_f = A})$

Vidíme, že $D_{f\circ g} = \{x \in B \,;\, g(x) \in A\}$
$= \{x \in \mathbb{D}_g : g(x) \in \mathbb{D}_f\}$







Může se stát, že $g(B) \cap A = \emptyset$.

Pak $f \circ g$ vůbec není def.

Například: • $\sqrt{-e^x}$

$g(x) = -e^x \qquad g: \mathbb{R} \longrightarrow \mathbb{R}$

$\mathbb{H}_g = g(\mathbb{R}) = (-\infty, 0) = g(B)$

$f(y) = \sqrt{y} \qquad \mathbb{D}_f = [0, \infty) = A$

$g(B) \cap A = \emptyset$.

VÍCE PROMĚNNÝCH:

Všimněme si, že $\mathbb{R}^2 \cap \mathbb{R}^3 = \emptyset$,

$\mathbb{R}^n \cap \mathbb{R}^m = \emptyset$, $n \neq m$.

$F: \mathbb{R}^2 \to \mathbb{R}^3$, $G: \mathbb{R}^2 \to \mathbb{R}^3$

Se nedají složit $G(\mathbb{R}^2) \subseteq \mathbb{R}^3 \cap D_F = \emptyset$.

$F: \mathbb{R}^k \to \mathbb{R}^l$, $G: \mathbb{R}^m \to \mathbb{R}^n$

$k, l, m, n \in \mathbb{N}$.

- $F \circ G$ dává smysl, pouze pokud $n = k$

$$F(\underbrace{G(\overbrace{x}^{\in \mathbb{R}^m})}_{\in \mathbb{R}^n})$$

- $G \circ F$ ... $l = m$.

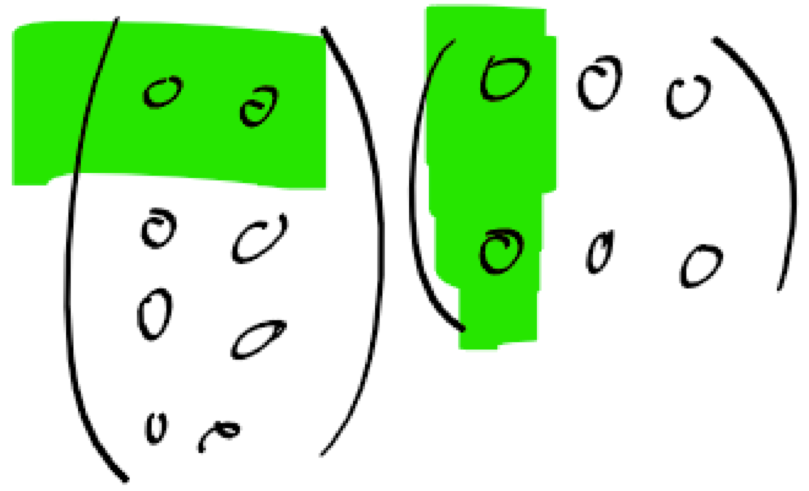
LIN. ALGEBRA

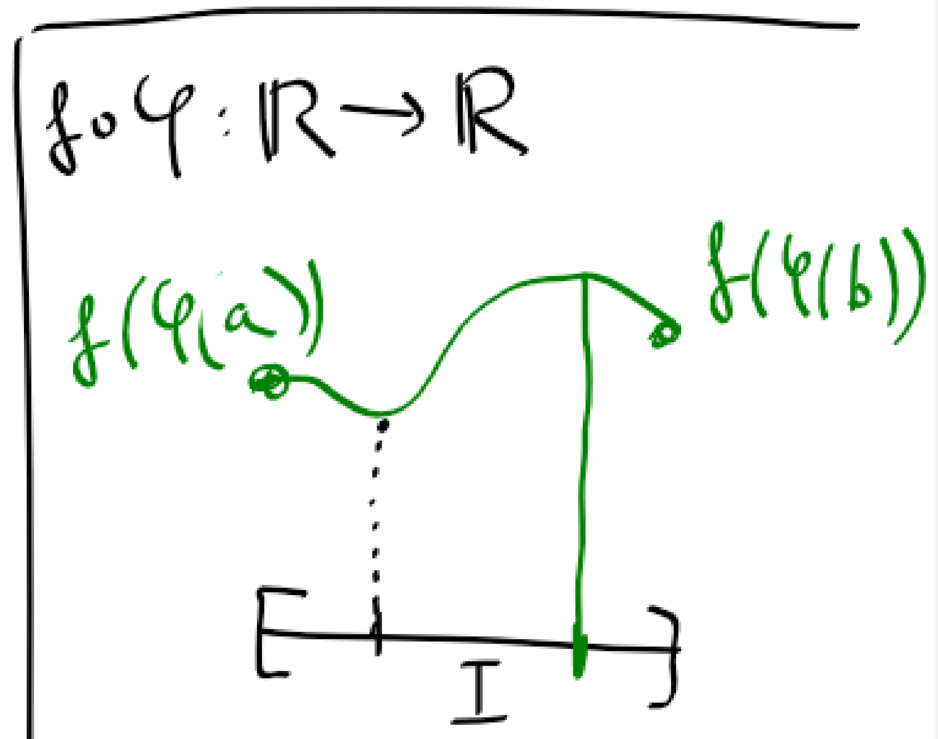
A: mce 2×3 ~ $\mathbb{R}^3 \to \mathbb{R}^2$

B: mce 4×2 ~ $\mathbb{R}^2 \to \mathbb{R}^4$

$B \cdot A$ je typu 4×3

$$\mathbb{R}^3 \xrightarrow{A} \mathbb{R}^2 \xrightarrow{B} \mathbb{R}^4$$





Speciálně nás zajímá situace:

Vnitřní: $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}^d$ $\quad (D_\varphi = I)$

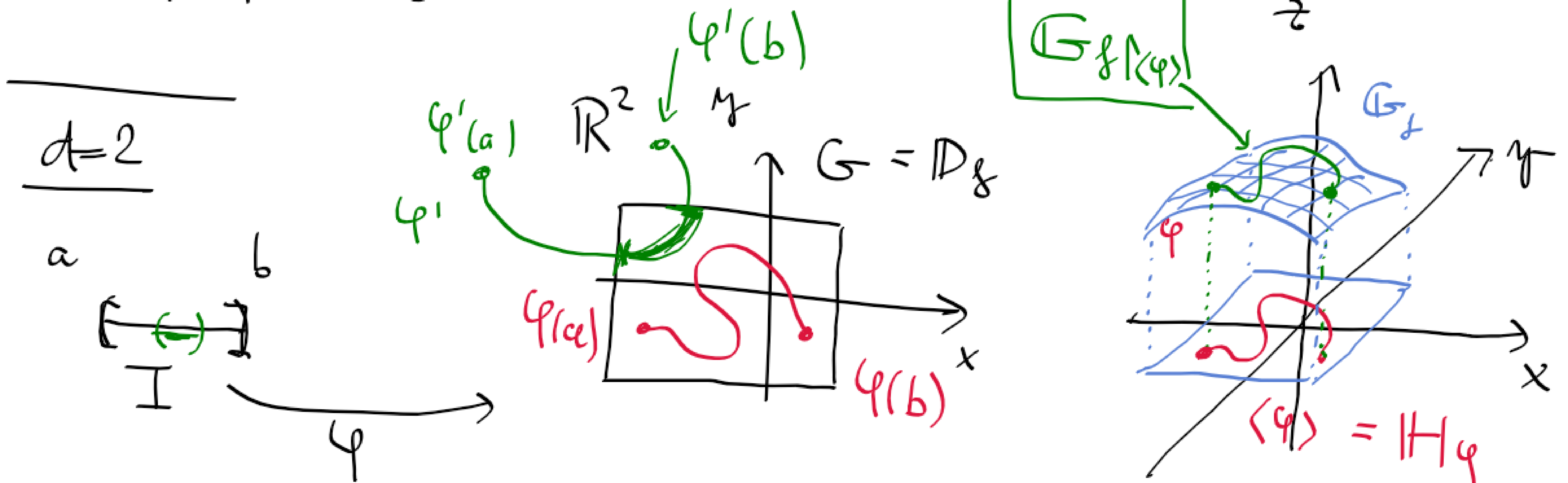
Vnější fce: $f : G \subseteq \mathbb{R}^d \to \mathbb{R}$ $\quad (D_f = G)$

Dává smysl formálně: $\mathbb{R} \xrightarrow{\varphi} \mathbb{R}^d \xrightarrow{f} \mathbb{R}$

- $f \circ \varphi$ ... $D_{f \circ \varphi} = ?$
- $\varphi \circ f$ $\quad (\mathbb{R}^d \xrightarrow{f} \mathbb{R} \xrightarrow{\varphi} \mathbb{R}^d)$. (nezajímá nás)

$$D_{f \circ \varphi} = \{ t \in I : \varphi(t) \in G \} \quad \boxed{f \circ \varphi}$$

$d = 2$



$\underline{d=2}$, φ je přímka

$f: \mathbb{R}^2 \to \mathbb{R}$, $\varphi(t) := a + t\vec{v} =$

$a, \vec{v} \in \mathbb{R}^2$

$$= (a_1, a_2) + t\cdot(v_1, v_2) =$$

$$= \underbrace{(a_1 + tv_1, a_2 + tv_2)}.$$

$f(x,y)$ složíme $\varphi(t)$:

$$f(\varphi(t)) = f(a + t\cdot\vec{v}) =$$

$$= f(a_1 + t\cdot v_1, a_2 + t\cdot v_2)$$

Jak vypadá derivace funkce $g(t) := \overbrace{f\circ\varphi(t)}^{f(\varphi(t))}$?

Řetízkové pravidlo:

$$\left[g'(t) = \frac{\partial f}{\partial x}(\varphi(t))\cdot \underbrace{\varphi_1'(t)}_{1} + \frac{\partial f}{\partial y}(\varphi(t))\cdot \underbrace{\varphi_2'(t)}_{0} \right]$$

Pro $v = \vec{e}_1$

Co když $\vec{v} \in \{\vec{e}_1, \vec{e}_2\}$ $\vec{e}_1 = (1,0)$, $\vec{e}_2 = (0,1)$.

$\underline{\vec{v} = \vec{e}_1}$:

$$f(\varphi(t)) = f(a_1 + t\cdot 1, a_2 + t\cdot 0) =$$

$$= f(a_1 + t, a_2) =: g_x(t)$$

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g_x je tzv. parciální funkce.

Její derivace je parciální D.:

$$\frac{\partial f}{\partial x}(\underbrace{a_1, a_2}_{a}) = g_x'(0) = \frac{d}{dt} f(a_1 + t, a_2)\Big|_{t=0}$$

$$g_x(\underline{0}) = f(a_1, a_2).$$

Podobně podle y.

<u>Příkládeček</u>: $f(x,y) = 100 - x^2 - y^2$

$\varphi(t) = (2,0) + t \cdot \vec{e_2}$ $\quad \vec{e_2} = (0,1) \in \mathbb{R}^2$.

$= (2,0) + t \cdot (0,1) = \underline{(2,t)}$.

$f \circ \varphi(t) = f(\varphi(t)) = f(2,t) = 100 - 2^2 - t^2$.

$f \circ \varphi(t) = 96 - t^2 \; =: g_y(t)$

$g_y'(t) = -2t \quad , \quad g_y'(2) = -4$

Jakému bodu v $\mathbb{R}^2$ ~~to~~ odpovídá?

$\varphi(2) = (2,2) \in \mathbb{R}^2$. Tj.

$$\left[\frac{\partial f}{\partial y}(2,2) = -4 \right].$$

<u>V PRAXI</u>: „y je konstanta"

$$\frac{\partial (x^2+y^2)}{\partial x} = 2x$$

$$\frac{\partial}{\partial y}(x^2+y^2) = 2y.$$

$$\frac{\partial f(6,0)}{\partial x} = 2 \cdot 6 = 12.$$

Příklad: Nalezněte max a min.

$f(x,y) = 2x + 8y$ na $M = \{(x,y) : \underbrace{x^2+2y^2=1}\}$

Parametrizace M je:

$$\varphi : \left[\begin{array}{l} x = \cos t \\ y = \frac{\sqrt{2}}{2} \sin t \end{array}\right. , \; t \in [0, 2\pi]$$

[Ov.: $\forall t \in [0, 2\pi] : \quad \underbrace{\varphi_1(t)^2 + 2\varphi_2(t)^2} = 1$]

Vlastně chceme extrémy $f \circ \varphi$.

$$f \circ \varphi(t) = f(\varphi(t)) = f\left(\cos t, \frac{\sqrt{2}}{2} \sin t\right) =$$

$$= 2 \overbrace{\cos t}^{x} + 8 \cdot \overbrace{\frac{\sqrt{2}}{2} \sin t}^{y}.$$

$$(f \circ \varphi)'(t) = -2 \sin t + 4\sqrt{2} \cos t = 0$$

$$2\sqrt{2} \cos t = \sin t \qquad \frac{\sin t}{\cos t} = 2\sqrt{2}$$

$$\varphi \left[\begin{array}{l} x = \cos t \\ y = p \cdot \sin t \end{array}\right.$$

$\varphi(0) = (\cos 0, p \cdot \sin 0) = (1, 0)$

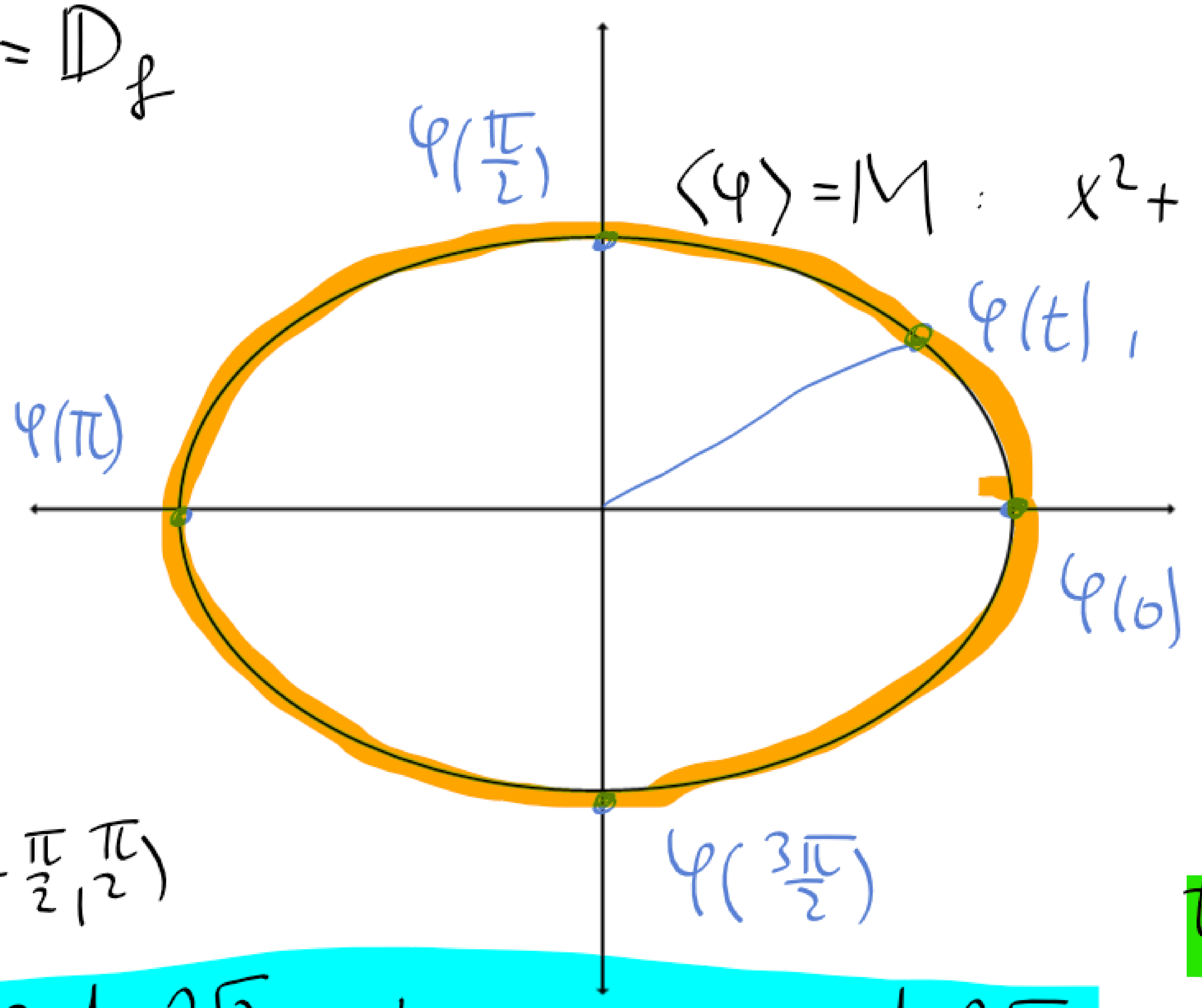
$\varphi(\frac{\pi}{2}) = (0, p) = (0, p)$

$t = 0$, $y = 0$ … $x^2 + 2y^2 = 1$

$x^2 = 1$ … $(\pm 1, 0)$

$t = \frac{\pi}{2}$, $x = 0$ … $2y^2 = 1$ … $(0, \pm \frac{\sqrt{2}}{2})$

$\mathbb{R}^2 = D_f$



$\in (-\frac{\pi}{2}, \frac{\pi}{2})$

$t_1 = \operatorname{arctg} 2\sqrt{2}$, $t_2 = \pi + \operatorname{arctg} 2\sqrt{2}$, $t_3 = 0$, $t_4 = 2\pi$

$f(\varphi(t_i)) = f(\cos t_i, \frac{\sqrt{2}}{2}\sin t_i)$ atd., $i = 1, \dots, 4$

Zjistíme, že max je v bodě t_1, min t_2

POZN.: $\varphi(0) = \varphi(2\pi) = (1,0) \in \mathbb{R}^2$.

$$f(\varphi(t_1)) = f\left(\tfrac{1}{3}, \tfrac{2}{3}\right) = 2\cdot\tfrac{1}{3} + 8\cdot\tfrac{2}{3} = \tfrac{18}{3} = \underline{\underline{6}}$$

$$f(\varphi(t_2)) = f(-\varphi(t_1)) = -f(\varphi(t_1)) = \underline{\underline{-6}}.$$

$$f(\varphi(\underbrace{t_3}_{0})) = f(\varphi(\underbrace{t_4}_{2\pi})) = f(1,0) = 2\cdot 1 + 8\cdot 0 = 2.$$

Vrstevnice funkce f: množiny bodů (x,y), kde f je konstantní $c \in \mathbb{R}$

$f(x,y) = c$ ... rovnice vrstevnice

$2x + 8y = c \implies y = \frac{c - 2x}{8} = \frac{c}{8} - \frac{1}{4}x$

Geometricky:

$\mathbb{R}^2$